PROPAGATION OF LAMINAR JETS OF COMPRESSIBLE FLUID ALONG A FLAT WALL

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The method of integral relations is applied to an approximate solution of dynamic and thermal problems in the propagation of laminar gas jets of small radial extent along a plane wall. In distinction from the work of Riley [1], all the basic characteristics of the flow and of heat transfer are presented in a form convenient for practical application over a rather wide range of variation of Prandtl number.

We shall suppose that an axisymmetric jet issues from a circular slit and propagates along a plane wall which coincides with the plane z = 0 (r is reckoned from the axis of symmetry of the jet). We shall represent the jet source as a slit between the plane z = 0 and a disk parallel to it.

The system of differential equations defining the above problem regarding propagation of a gas jet along a wall, has the form, in dimensionless coordinates,

$$\rho\left(V_r\frac{\partial V_r}{\partial r}+V_z\frac{\partial V_r}{\partial z}\right)=\frac{\partial}{\partial z}\left(\mu\frac{\partial V_r}{\partial z}\right),$$
$$\frac{\partial}{\partial r}(\rho rV_r)+\frac{\partial}{\partial z}(\rho rV_z)=0,$$
$$\rho\left(V_r\frac{\partial i}{\partial r}+V_z\frac{\partial i}{\partial z}\right)=\frac{1}{\sigma}\frac{\partial}{\partial z}\left(\mu\frac{\partial i}{\partial z}\right)+$$
$$+(k-1)M_\infty^2\mu\left(\frac{\partial V_r}{\partial z}\right)^2,$$
$$\rho=\frac{1}{i},\quad \mu=i^n,\quad i=IC_pT. \tag{1}$$

The boundary conditions may be written as

$$V_r=0,\ V_z=0,\ i=i_w\ \text{at}\ z=0,$$
$$V_r=0\ \text{at}\ z=\delta,$$
$$i=0\ \text{at}\ z=\delta_T, \tag{2}$$

where δ is the thickness of the wall layer of the jet and δ_T is the thickness of the thermal layer.

The condition that there be a non-trivial solution to the hydrodynamic problem, as is known [2], has the form

$$\frac{d}{dr}\int_0^\delta \rho rV_r\left(\int_0^z \rho rV_r^2dz\right)dz+\int_0^\delta \rho\mu r^2V_r\frac{\partial V_r}{\partial z}dz=0. \tag{3}$$

It is clear that for $\rho\mu$ = const the second term on the left of this relation goes to zero. In fact, since $i\rho = 1$, the product $\rho\mu\mu^{(1-n)/n}$ will be constant. But, if we take into account that $0.74 \le n \le 1$, then, over the whole range of change, we may replace the quantity $\mu^{(1-n)/n}$ by the mean value

$$\overline{\mu}^{(1-n)/n}=\frac{1}{\mu}\int_0^\mu \mu^{(1-n)/n}d\mu. \tag{4}$$

The mean deviation from the condition $\rho\mu$ = const will then not exceed 10%.

In view of this, the condition for a non-trivial solution of the hydrodynamic problem for a radial jet of variable density ρ may be assigned in the form

$$E=\int_0^\delta \rho rV_r\left(\int_0^z \rho rV_r^2dz\right)dz=\text{const}. \tag{5}$$

We shall write the first and third equations of system (1), with the aid of the continuity equation, in the form

$$\frac{\partial}{\partial r}(\rho rV_r^2)+\frac{\partial}{\partial z}(\rho rV_rV_z)=\frac{\partial}{\partial z}\left(\mu r\frac{\partial V_r}{\partial z}\right),$$
$$\frac{\partial}{\partial r}(rV_r)+\frac{\partial}{\partial z}(rV_z)=\frac{1}{\sigma}\frac{\partial}{\partial z}\left(\mu r\frac{\partial i}{\partial z}\right)+$$
$$+(k-1)M_\infty^2\mu r\left(\frac{\partial V_r}{\partial z}\right)^2. \tag{6}$$

Carrying out integration of these equations with respect to z, we arrive at the integral relations for a radial gas jet in the form

$$\frac{d}{dr}\int_0^\delta \rho rV_r^2dz=-\mu_w r\left[\frac{\partial V_r}{\partial z}\right]_{z=0},$$
$$\frac{d}{dr}\int_0^{\delta_T} rV_rdz=-\frac{\mu_w r}{\sigma}\left[\frac{\partial i}{\partial z}\right]_{z=0}+$$
$$+(k-1)M_\infty^2 r\int_0^{\delta_T}\mu\left(\frac{\partial V_r}{\partial z}\right)^2dz. \tag{7}$$

The velocity profile and the change in enthalpy in the sections of the jet will be given by the polynomials

$$V_r=V_{\max}F(\eta),$$
$$i=i_wf(\eta_T), \tag{8}$$

where $\eta = z/\delta$, $\eta_T = z/\delta_T$.

The choice of the polynomial coefficients $F(\eta)$ and $f(\eta_T)$ will satisfy the boundary conditions

$$F=0\ \text{at}\ \eta=0\ \text{and at}\ \eta=1,$$
$$F=1,\ \frac{\partial F}{\partial\eta}=0\ \text{at}\ \eta=\frac{1}{2}, \tag{9}$$

$$f=0,\ \frac{\partial f}{\partial\eta_T}=0,\ \frac{\partial^2 f}{\partial\eta_T^2}=0\ \text{at}\ \eta_T=1,$$
$$f=1\ \text{at}\ \eta_T=0. \tag{10}$$

Then the maximum velocity in terms of the mass flow rate of fluid will be determined by the formula

$$V_{\max}=G/2\pi rA_1\rho\delta, \tag{11}$$

where

$$A_1 = \int_0^1 F(\eta)\, d\eta. \qquad (12)$$

But if we take into account that

$$\rho = 1/i_w f(\eta_T) = \rho_w / f(\eta_T), \qquad (13)$$

we obtain

$$V_{\max} = G f(\eta_T) / 2\pi r A_1 \rho_w \delta. \qquad (14)$$

Then the desired velocity profile will take the form

$$V_r = GF(\eta) f(\eta_T)/2\pi r A_1 \rho_w \delta. \qquad (15)$$

Taking into consideration that

$$\left.\frac{\partial V_r}{\partial z}\right|_{z=0} = \frac{GF'(0)}{2\pi r A_1 \rho_w \delta^2}, \qquad (16)$$

we shall give the momentum equation the form

$$\frac{d}{dr}\left(\frac{G^2}{r\delta}\right) = -2\pi A_0 A_1 \mu_w \frac{G\Delta^3}{\delta^2 H_1(\Delta)}, \qquad (17)$$

where

$$\Delta = \delta_T/\delta, \qquad (18)$$

$$A_0 = F'(0), \qquad (19)$$

$$H_1(\Delta) = \Delta^3 \int_0^1 F^2(\eta) f(\eta_T)\, d\eta_T. \qquad (20)$$

If we use condition (5), we find that

$$r\delta = G^3 E_1/(2\pi A_1)^3 \rho_w E, \qquad (21)$$

where

$$E_1 = \int_0^1 F^2(\eta) f(\eta_T) \left[\int_0^{\eta} F(\eta)\, d\eta\right] d\eta. \qquad (22)$$

Then the differential equation to determine the flow rate of fluid takes the form

$$G^5 \frac{d}{dr}\left(\frac{1}{G}\right) = -(2\pi A_1)^4 \rho_w \mu_w A_0 \Delta^3 \frac{Er^2}{E_1 H_1(\Delta)}. \qquad (23)$$

Carrying out integration of this equation, with the condition that the enthalpy i_W does not depend on r, we obtain

$$G = 2\pi A_1 \Delta \sqrt[4]{4\rho_w \mu_w A_0 E r^3 / 3E_1 \Delta H_1(\Delta)}. \qquad (24)$$

This allows us to find the values of all the basic flow parameters. Making the calculation, we obtain

$$\delta = \frac{4}{3} \frac{\mu_w \Delta^2}{H_1(\Delta)} \sqrt[4]{\frac{3A_0^3 E_1 \Delta H_1(\Delta) r^5}{4\rho_w \mu_w E}}, \qquad (25)$$

$$V_{\max} = \frac{3i}{4\Delta^2} \sqrt{\frac{4\rho_w E \Delta H_1(\Delta)}{3\mu_w A_0 E_1 r^3}}, \qquad (26)$$

$$\tau_w = \frac{3}{4} \frac{H_1(\Delta)}{\Delta^4} \sqrt[4]{\frac{3E^3 \Delta H_1(\Delta)}{4\rho_w \mu_w A_0 E_1^3 r^{11}}}, \qquad (27)$$

$$K = 2\pi \frac{H_1(\Delta)}{\Delta^4} \sqrt[4]{\frac{3E^3 \Delta H_1(\Delta)}{4\rho_w \mu_w A_0 E_1^3 r^3}}, \qquad (28)$$

$$GK = 4\pi^2 A_1 \frac{EH_1(\Delta)}{E_1 \Delta^3}, \qquad (29)$$

$$G_f = 4\pi \mu_w A_0 r / G A_1. \qquad (30)$$

It is evident that for practical use of formulas (24)–(30) it is necessary in addition to the flow constants, to give the Prandtl number values.

If the jet source is a slit of height δ_a between the planes z = 0 and a disk of radius $r = a$ parallel to it, then, assuming that the stream through the slit is uniform with velocity V_a and enthalpy i_a, we obtain

$$E = \frac{1}{2} \frac{V_a^3 \delta_a^2 a^2}{i_a^2}. \qquad (31)$$

We shall now turn to determination of the thickness of the thermal layer. To do this we use the heat balance equation, which, following substitution of the profiles of velocity and enthalpy, takes the form of the Bernoulli differential equation

$$\frac{di_w}{dr} + \frac{M}{r} i_w + \frac{N}{r^4} i_w^{1-n} = 0, \qquad (32)$$

where

$$M = \frac{3}{3+n}\left(1 - \frac{BH_1(\Delta)}{\sigma \Delta^6 H(\Delta)}\right); \qquad (33)$$

$$N = \frac{9(1-k) M_\infty^2 E H_1^2(\Delta) H_2(\Delta)}{4(3+n) A_0^2 E \Delta^9 H(\Delta)}; \qquad (34)$$

$$B = -\frac{f'(0)}{A_0} \int_0^1 F^2(\eta)\, d\eta; \qquad (35)$$

$$H(\Delta) = \frac{1}{\Delta} \int_0^1 F(\eta) f(\eta_T)\, d\eta_T; \qquad (36)$$

$$H_2(\Delta) = \int_0^1 f^n(\eta_T) \left[F(\eta) f(\eta_T)\right]'_{\eta_T} d\eta_T. \qquad (37)$$

If we make the substitution

$$u(r) = i_w^r, \qquad (38)$$

we obtain the equation

$$\frac{du}{dr} + \frac{nM}{r} u + \frac{nN}{r^4} = 0, \qquad (39)$$

which has the general solution

$$u = r^{-nM}(C - nNr^{nM-3}/nM - 3). \qquad (40)$$

Therefore,

$$i_w = r^{-M} \sqrt[n]{C - nNr^{nM-3}/(nM-3)}. \qquad (41)$$

If we assume that nM ≠ 3 and $i_W = i'_W$ at $r = a = 1$, we find

$$C = i_w'^n + nN/(nM-3). \qquad (42)$$

Then, to determine δ_T, we obtain the equation

$$i_w^n = \frac{i_w'^n}{r^{nM}} + \frac{nN}{(nM-3) r^{nM}} (1 - r^{nM-3}), \qquad (43)$$

or

$$\frac{\mu_w r^{nM} - \mu_w'}{1 - r^{nM-3}} = \frac{3n(k-1) M_\infty^2 \sigma E H_1^2(\Delta) H_2(\Delta)}{4A_0^2 E_1 [3\sigma \Delta^3 H(\Delta) - nBH_1(\Delta)]}, \qquad (44)$$

where μ_W' is the value of the viscosity at the wall, at $r = a = 1$.

Basic Flow and Heat Transfer Parameters

Parameters	for $\sigma=0.536$	for $\sigma=0.72$	for $\sigma=1$	D
V_{max}	$0.13D$	$0.15D$	$0.19D$	$i\sqrt{\frac{\rho_w E}{\mu_w E_1 r^3}}$
G	$11.47D$	$10.83D$	$10.39D$	$\sqrt[4]{\frac{\rho_w \mu_w E r^3}{E_1}}$
K	$0.22D$	$0.28D$	$0.36D$	$\sqrt[4]{\frac{E^3}{\rho_w \mu_w E_1^3 r^3}}$
δ	$20.4D$	$16.77D$	$14.73D$	$\sqrt[4]{\frac{\mu_w^3 E_1 r^5}{\rho_w E}}$
τ_w	$0.025D$	$0.034D$	$0.044D$	$\sqrt[4]{\frac{E^3}{\rho_w \mu_w E_1^3 r^{11}}}$
C_f	$6.4D$	$6.9D$	$7.2D$	$\sqrt[4]{\frac{\mu_w^3 E_1 r}{\rho_w E}}$
GK	$0.25D$	$0.31D$	$0.39D$	$\pi^2 \frac{E}{E_1}$
$Nu_{r,w}$	$0.14D$	$0.148D$	$0.15D$	$\sqrt[4]{\frac{\rho_w E}{\mu_w^3 E_1 r^5}}$
$C_f Nu_{r,w}$	0.9	1.08	1.1	—

If there is no energy dissipation, then putting N = = 0 in (43), we obtain

$$i_w = i'_w r^{-M}, \tag{45}$$

i. e., the equation for determining δ_T takes the form

$$\Delta^6 H(\Delta)/H_1(\Delta) = B \ln r/\sigma \ln [r \sqrt[3]{(i_w/i'_w)^{3+n}}]. \tag{46}$$

In the case when the wall temperature is constant, i.e., $i_W = i'_W$ = const, we obtain in place of the differential equation (32), an equation to determine δ_T in the form

$$\frac{M}{r} i_w + \frac{N}{r^4} i_w^{1-n} = 0 \tag{47}$$

or

$$\Delta^6 \frac{H(\Delta)}{H_1(\Delta)} = \frac{B}{\sigma} + \frac{3(k-1) M_\infty^2 E H_1(\Delta) H_2(\Delta)}{4A_0^2 \mu_w E_1 r^3}. \tag{48}$$

Hence, we see that allowance for energy dissipation is of appreciable importance only in the neighborhood of the point of efflux of the jet along the wall. At sufficiently large values of $r \gg R_0$, as determined experimentally, the second term on the right side of (48) drops out, and we obtain

$$\Delta^6 H(\Delta)/H_1(\Delta) = B/\sigma. \tag{49}$$

This equation allows us to find δ_T for the problem of flow over a wall of a compressible fluid with constant wall temperature and in the absence of energy dissipation.

Knowing Δ, let us determine the heat transfer coefficient, and the Nusselt thermal similarity parameter, referred to the distance r from the source, from the formulas

$$\alpha_w = \frac{\lambda_w}{\delta_T} f'(0), \tag{50}$$

$$\mathrm{Nu}_{r,w} = -\frac{3f'(0)}{4\Delta^3} \sqrt[4]{\frac{4\rho_w E H_1^3(\Delta)}{3\mu_w^3 A_0^3 \Delta E_1 r}}. \tag{51}$$

We note that in flow along a wall of a jet of compressible fluid, the relation

$$C_f \mathrm{Nu}_{r,w} = -\frac{3f'(0) H_1(\Delta)}{2A_1^2 \Delta^4} = \mathrm{const} \tag{52}$$

will hold.

If the velocity field is approximated by a second degree polynomial,

$$F(\eta) = 4\eta(1-\eta), \tag{53}$$

and the temperature field by a third degree polynomial,

$$f(\eta_T) = (1-\eta_T)^3, \tag{54}$$

then the basic characteristics of the flow and of heat transfer, for Prandtl number values $\sigma = 0.536$, 0.72, and 1, in the case of constant wall temperature, and in the absence of energy dissipation, will be determined by the formulas given in the table.

We see from the table that variation of Prandtl number in the range 0.536 to 1 has practically no effect on C_f, and even less effect on the Nusselt number. Therefore, if the effect of Prandtl number is neglected, the calculations are simplified even more.

REFERENCES

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2. L. G. Loitsyanskii, The Laminar Boundary Layer [in Russian], Fizmatgiz, 1962.

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